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1996 J. Phys. A: Math. Gen. 29 3903

(http://iopscience.iop.org/0305-4470/29/14/015)

J. Phys. A: Math. Gen. **29** (1996) 3903–3910. Printed in the UK

Non-universal spectral rigidity of quantum pseudo-integrable billiards

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Received 28 September 1995

Abstract. We obtain the Dyson–Mehta Δ_3-statistic for pseudo-integrable billiards and show that it is non-universal with a universal trend, also that this trend is similar to the one for integrable billiards. We present a formula, based on exact semiclassical calculations and the proliferation law of periodic orbits, which gives rigidity for the entire range of L. To consolidate our theory, we discuss several examples finding complete agreement with the numerical results, and also the underlying fundamental reasons for the non-universality.

One of the central issues of quantum chaos is the universality of spectral fluctuations in integrable and chaotic limits of quantum dynamical systems. It has become clear through the semiclassical analysis of these Hamiltonian dynamical systems, that the source of universality is the nature of the periodic orbits, and also that the same source is responsible for the non-universality in spectral fluctuations. We believe that an exact semiclassical analysis of the systems that admit non-universal spectral fluctuations can help give a deeper understanding of universality and its breakdown. In this paper, we treat this dichotomy by considering dynamical systems at the 'edge of chaos', the pseudo-integrable billiards [1–3], wherein a particle moves freely inside a given enclosure, reflecting specularly from the walls. For these systems the generating function (first integral) which is independent of the Hamiltonian H and in involution with H, does not exist on a countable set of singular points. Due to the mathematical intractability of these systems, few exact results are known [4, 5]. Numerical studies performed on the energy spectra of these systems [6–12] led to the belief that the spectral statistics is intermediate between those for the Poisson ensemble and the Gaussian orthogonal ensemble (GOE) of random matrices. However, the results on various measures of the spectral statistics as analysed from the periodic orbit theory have been indicative only and do not bring out a complete or explicit picture of the underlying correlations. The aim of this paper is to provide quantitative as well as qualitative statements about the spectral rigidity of integrable and pseudo-integrable billiards (IB and PIB), inspired by examples which can be solved exactly, and compare the results with those of [7, 8, 13]. We employ the semiclassical formalism [14] in conjunction with the law of proliferation of periodic orbits derived in [5] and establish non-universality of the Dyson–Mehta statistic for these systems.

In their statistical theory of energy levels of complex systems, Dyson and Mehta [15] proposed the Δ-statistic to study spectral fluctuations on the intermediate energy scale, the

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0305-4470/96/143903+08$19.50

most popular being the Δ_3-statistic defined as a local average of the mean-square deviation of the spectral staircase from the best-fit straight line over an energy range corresponding to L mean level spacings. This can be expressed in terms of the contributions of periodic orbits originating in semiclassical expansion of the oscillatory correction $d_{\rm osc}(E)$ to the mean density of states [14]. For the two-dimensional PIB, the $d_{\rm osc}(E)$ can be written as [2]†

$$d_{\rm osc}(E) = \frac{1}{4\pi\hbar^2}\sum_{r,j} A_j J_0\left(\frac{\sqrt{E}\ell_{j,r}}{\hbar}\right) \tag{1}$$

where j represents the label for primitive periodic orbits (PPO) and r is the label for repetitions, mass is taken as $\frac{1}{2}$. A_j denotes the projective phase space areas of the bands in which periodic orbits occur. It can be shown using semiclassical formalism of [14] that the rigidity is given by

$$\Delta_3(L,E) = \frac{E^{1/2}}{4\pi^3\hbar}\sum_j\sum_k \frac{A_jA_k}{\ell_j^{3/2}\ell_k^{3/2}}\cos\left\{\frac{\sqrt{E}(\ell_j-\ell_k)}{\hbar}\right\}G(y_j,y_k) \tag{2}$$

where $G(y_j, y_k) = f(y_j - y_k) - f(y_j)f(y_k) - 3f'(y_j)f'(y_k)$, and $f(y) = \sin y/y$, $y = L\ell/(4\langle d\rangle E^{1/2}\hbar)$ (ℓ is the length of the PO, $\langle d\rangle$ is the average density of states, $A_{\rm R}/4\pi\hbar^2$, $A_{\rm R}$ being the area of the enclosure).

The main questions that ensue from the numerical studies [6–8, 13] are: (i) are the levels of PIB uncorrelated and do they mimic a Poisson process over a certain range of L, as seen in IB?; (ii) is there any saturation of rigidity in PIB if L exceeds the system-dependent range, as seen for the IB?; (iii) what is the essential difference between PIB, IB and chaotic billiards in terms of level correlations?; (iv) can we obtain a formula for the rigidity such that Poisson and non-Poisson results follow in a natural way for the IB and PIB, respectively? This paper answers all these questions to a large (sometimes complete) extent. This success holds due to the fact that the rigidity is a direct consequence of the proliferation law with some simple, non-trivial modifications in the known formalism shown below.

Recalling equation (2), we employ the uniformity principle [16] and retain also, apart from the diagonal, the off-diagonal part corresponding to the systematic degeneracies in the lengths of POs (giving the same contribution to the rigidity as the diagonal terms). From the exact results on some of the PIBs [4], one can classify the bands of the POs in such a way that the projective phase space area occupied by all periodic orbits in a given class (say α) is identical (A_α). It seems that such classification of the bands in terms of the projective phase space areas is also possible for the generic PIBs about which we will comment later. With this in mind, we can write $\Delta_3(L,E)$ as

$$\begin{aligned}\Delta_3(L,E) = \frac{E^{1/2}}{4\pi^3\hbar}\Bigg[&\sum_\alpha g_\alpha^2A_\alpha^2\sum_j\frac{G(y_{\alpha,j})}{\ell_{\alpha,j}^3}\\ &+\sum_\alpha\sum_\beta(1-\delta_{\alpha,\beta})g_\alpha g_\beta A_\alpha A_\beta\sum_j\sum_k\delta_{\ell_{\alpha,j},\ell_{\beta,k}}\frac{G(y_{\alpha,j},y_{\beta,k})}{\ell_{\alpha j}^{3/2}\ell_{\beta k}^{3/2}}\Bigg]\end{aligned} \tag{3}$$

where $G(y) = 1 - f^2(y) - 3(f'(y))^2$ and Greek subscripts denote classes of periodic bands. In equation (3), $g_\alpha(g_\beta)$ denotes the number of POs with the same action belonging to the class $\alpha(\beta)$, and $\delta_{i,j}$ is the usual Kronecker symbol. In equation (3) the summation $\sum_j G(y_{\alpha,j})/\ell_{\alpha,j}^3$ can be written as $\int_{y_{\rm min}}^\infty {\rm d}F_\alpha\, G(y_\alpha)/\ell_\alpha^3$ in the continuum limit due to the mathematical nature of the summand, where ${\rm d}F_\alpha$ represents the number of periodic orbits

† For most of the PIB known today, periodic orbits are of neutral or marginal stability and hence occur in bands. Equation (1) is only valid for such cases.

within length ℓ and $\ell + d\ell$. This dF_α can be deduced from the proliferation law (average or asymptotic part) [5], which gives the average number of periodic orbits of length $\leqslant \ell$ as (for different classes α)

$$F_\alpha(\ell) = a_\alpha \ell^2 + b_\alpha \ell + c_\alpha. \tag{4}$$

We emphasize that this law not only arises from enumeration of periodic orbits [5] but also can be deduced from the eigenvalue spectrum by inverting the trace formula analytically [17]. It was shown in [17] that (4) is the classical analogue of the celebrated Weyl formula. It may be noted that the exact proliferation law will also contain an oscillatory term $dF_{\alpha\mathrm{osc}}$. We neglect the effect of this term since its contribution to the above integral will be extremely small due to the oscillatory nature of $dF_{\alpha\mathrm{osc}}$ around zero. With equation (4), we have now (after unfolding the spectrum via rescaled energies $\mathcal{E} = E\langle d\rangle$

$$\begin{aligned}\Delta_3(L,\mathcal{E}) = {} & \frac{L}{2\pi^2 A_{\mathrm{R}}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 a_\alpha I_{1,\alpha} + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta a_\eta I_{1,\eta}\right] \\ & + \frac{L^2}{8\pi^{3/2} A_{\mathrm{R}}^{3/2}\mathcal{E}^{1/2}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 b_\alpha I_{2,\alpha} + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta b_\eta I_{2,\eta}\right]\end{aligned} \tag{5}$$

where

$$I_{1,\alpha} = \int_{y_{\min,\alpha}}^\infty dy_\alpha\, y_\alpha^{-2} G(y_\alpha) \qquad I_{2,\alpha} = \int_{y_{\min,\alpha}}^\infty dy_\alpha\, y_\alpha^{-3} G(y_\alpha) \tag{6}$$

both these can be evaluated easily with $y_{\min} = \pi L/L_{\max}$, where $L_{\max,\alpha} = \sqrt{4\pi A_{\mathrm{R}}\mathcal{E}/\ell_{\min,\alpha}^2}$. For small $y_{\min}$, $I_1 = 2\pi/15$, $I_2 = \frac{1}{9}$; and for large $y_{\min}$, $I_{1,\alpha} = L_{\max,\alpha}/\pi L$, $I_{2,\alpha} = L_{\max,\alpha}^2/2\pi^2 L^2$. For $L < \min_\alpha L_{\max,\alpha}/\pi$,

$$\begin{aligned}\Delta_3(L,\mathcal{E}) = {} & \frac{L}{15\pi A_{\mathrm{R}}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 a_\alpha + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta a_\eta\right] \\ & + \frac{L^2}{72\pi^{3/2} A_{\mathrm{R}}^{3/2}\mathcal{E}^{1/2}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 b_\alpha + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta b_\eta\right].\end{aligned} \tag{7}$$

For $L \gg \max_\alpha L_{\max,\alpha}/\pi$,

$$\begin{aligned}\Delta_3(L,\mathcal{E}) = {} & \frac{1}{2\pi^3 A_{\mathrm{R}}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 a_\alpha L_{\max,\alpha} + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta a_\eta L_{\max,\eta}\right] \\ & + \frac{1}{16\pi^{7/2} A_{\mathrm{R}}^{3/2}\mathcal{E}^{1/2}}\left[\sum_\alpha g_\alpha^2 A_\alpha^2 b_\alpha L_{\max,\alpha}^2 + \sum_\gamma\sum_\eta \delta_{\ell_\gamma,\ell_\eta} g_\gamma g_\eta A_\gamma A_\eta b_\eta L_{\max,\eta}^2\right].\end{aligned} \tag{8}$$

It is important to note that minimum and maximum (over α) $L_{\max}$ correspond, respectively, to the longest and shortest (over α) orbits of the set containing the shortest periodic orbits of different αs. It is the consequence of this observation that will lead us to understand the fundamental distinction between the spectral correlations of integrable and pseudo-integrable billiards.

Ignoring I_2 for the sake of brevity, the formula valid for the entire range of L is given by (denoting $L_{\max,\alpha}/\pi L$ by Λ_α),

$$\begin{aligned}\Delta_3(L,\mathcal{E}) = {} & \frac{L}{2\pi^2 A_{\mathrm{R}}}\sum_\alpha g_\alpha^2 A_\alpha^2 a_\alpha [\Lambda_\alpha - \tfrac{2}{3}\Lambda_\alpha^3 - \tfrac{3}{10}\Lambda_\alpha^5 - \tfrac{2}{15}\Lambda_\alpha\cos(2/\Lambda_\alpha) \\ & - \tfrac{1}{15}\Lambda_\alpha^2\sin(2/\Lambda_\alpha) + \tfrac{1}{15}\Lambda_\alpha^3\cos(2/\Lambda_\alpha) + \tfrac{3}{15}\Lambda_\alpha^4\sin(2/\Lambda_\alpha) \\ & + \tfrac{3}{10}\Lambda_\alpha^5\cos(2/\Lambda_\alpha) - \tfrac{4}{15}\,\mathrm{si}(2/\Lambda_\alpha)]\end{aligned} \tag{9}$$

where $\mathrm{si}(x) = -\int_x^\infty \mathrm{d}t\, t^{-1} \sin t$. Equation (9) along with the limiting results (7), (8); which clearly establishes a relation between Δ_3 and information about the classical periodic orbits such as proliferation law, band areas, degeneracies in lengths etc. This is the main result of our paper as it applies to all integrable and pseudo-integrable billiards. To understand the formulae better, we now propose to examine some paradigm systems carefully, and subsequently compare the results with the known numerical results.

In this regard, we consider the specific examples of an incommensurate rectangle billiard (IRB), the graveyard billiard (GB) and the $\pi/3$-rhombus billiard (RHB).

First, let us consider the IRB with sides $(\mathcal{L}, \gamma\mathcal{L})$, γ being an irrational number. All periodic orbits fall in a single class occupying projective phase space area $4A_{\mathrm{R}}$ ($A_{\mathrm{R}} = \gamma\mathcal{L}^2$), except the two shortest periodic orbit bands parallel to either pair of sides of IRB. The area of these two shortest periodic bands is $2A_{\mathrm{R}}$. The proliferation law for the IRB can be easily found by employing the ideas of stacking and replication [4]; we get (counting all the repetitions of the PPOs),

$$F_{\mathrm{IRB}}(\ell) = a\ell^2 + b\ell = \frac{\pi}{16\gamma\mathcal{L}^2}\ell^2 + \frac{\gamma+1}{4\gamma\mathcal{L}}\ell. \tag{10}$$

Using a, A $(= 4A_{\mathrm{R}})$ in (9) we get complete quantitative agreement with results obtained earlier [13] for $L < L_{\max}/\pi$. As observed in [13], the oscillations in Δ_3(L) are rather weak beyond the 'crossover regime'.

To get the correct saturation values of $\Delta_3(L)$, we have to consider the $\mathrm{O}(\ell)$ term in (10). Taking account of this, in the region where $L < L_{\max}/\pi$, we get

$$\Delta_3(L, \mathcal{E}) = \frac{L}{15} + \frac{1}{9\sqrt{2}\pi^{3/2}}\left(\frac{\gamma+1}{\gamma}\frac{L^2}{\mathcal{E}^{1/2}}\right). \tag{11}$$

The second term is quite small compared to the first one due to the factor $\mathcal{E}^{-1/2}$. For $L \gg L_{\max}/\pi$, on the other hand, we have

$$\Delta_3(L, \mathcal{E}) = \frac{\mathcal{E}^{1/2}}{2\pi^{3/2}}\left(1 + \frac{\sqrt{\gamma}(\gamma+1)}{2\pi}\right) \tag{12}$$

which is in very good agreement with the numerical results.

Our next example is the graveyard billiard, which is a simple variation of the barrier billiard described in [19]. We consider a linear barrier of length equal to half the side length $\mathcal{L}$ of the rectangle $(\mathcal{L}, 2\mathcal{L})$ placed at its centre. This is an example of a PIB whose invariant surface is topologically equivalent to a sphere with two handles (genus, $g = 3$). The law of proliferation is the same as $a_\alpha\ell^2 + b_\alpha\ell$. We have to obtain a_α, b_α for different classes of bands in this system. These calculations can be done by invoking ideas similar in spirit to those applied to the IRB. However, the presence of a barrier generates an infinite lattice of barriers and gaps with barrier-to-gap ratio as well as the distance between the two barriers placed on top of each other as unity. Thus, the end points of the barriers form lattice points which can be labelled by integer pairs entailing, thereby, a natural classification in terms of the coprime pairs (q, p). Each trajectory from origin to the coprime pairs (q, p) gives primitive periodic orbit closing at (mq, mp), where m is integer. The length of PPO is then given by $m\mathcal{L}\sqrt{(q^2 + p^2)}$. This classification leads to results that are summarized in table 1

With these, for $L < \min_\alpha(L_{\max,\alpha}/\pi) = \sqrt{\mathcal{E}/4\pi}$,

$$\Delta_3(L, \mathcal{E}) = \frac{L}{15} + \frac{1}{18\sqrt{2\pi^3}}\frac{L^2}{\mathcal{E}^{1/2}} \tag{13}$$

and for $L \gg \max_\alpha(L_{\max,\alpha}/\pi) = \sqrt{8\mathcal{E}/\pi}$,

$$\Delta_3(L, \mathcal{E}) \sim \left[\frac{(\sqrt{2}+13)}{12\pi^{3/2}} + \frac{9}{8\sqrt{2\pi^5}}\right]\mathcal{E}^{1/2}. \tag{14}$$

Table 1. Summary of results for GB.

Class	Closing point	Band area	Coeff. a	Coeff. b	Degeneracy g	$\ell_{\min}$
odd–odd	$(4q, 4p)$	$2A_{\mathrm{R}}$	$\pi/192\mathcal{L}^2$	0	2	$4\sqrt{2}\mathcal{L}$
even–odd[1]	$(2q, 2p)$	A_{R}	$\pi/48\mathcal{L}^2$	$1/12\mathcal{L}$	2	$\mathcal{L}$
even–odd[2]	$(2q, 2p)$	A_{R}	$\pi/48\mathcal{L}^2$	$1/12\mathcal{L}$	2	$\mathcal{L}$
odd–even	$(4q, 4p)$	$2A_{\mathrm{R}}$	$\pi/192\mathcal{L}^2$	$1/24\mathcal{L}$	2	$4\mathcal{L}$

We will discuss these results after we present calculations for yet another well-studied system—the $\pi/3$-rhombus billiard. This is an almost-integrable system with an invariant integral surface of genus two. For this system tesselletion of the plane is not complete and results in more general barrier structure [3]. We reproduce here, our results [4] about classification and distribution of periodic orbits. Again, here each trajectory from origin to a coprime pairs (q, p) represents PPO ending at $c(q, p)$. The length of the periodic orbit is given by $c\mathcal{L}\sqrt{(q^2 + 3p^2)}$ and A_{R} is $\sqrt{3}\mathcal{L}^{\in}/2$. Other important results are summarized in table 2.

Table 2. Summary of results for $\pi/3$-rhombus billiards.

Type	Class	Closing point	Band area	Coeff. a	Degeneracy g	$\ell_{\min}$
Centre–centre	odd–odd	$(q/2, p/2)$	A_{R}	$\pi/27A_{\mathrm{R}}$	2	$\sqrt{3}\mathcal{L}$
(q, p)	$2A_{\mathrm{R}}$	$\pi/108A_{\mathrm{R}}$	2	$2\sqrt{3}\mathcal{L}$		
	odd–even	(q, p)	A_{R}	$\pi/108A_{\mathrm{R}}$	2	$\sqrt{21}\mathcal{L}$
$(2q, 2p)$	$2A_{\mathrm{R}}$	$\pi/432A_{\mathrm{R}}$	2	$2\sqrt{21}\mathcal{L}$		
	even–odd	(q, p)	A_{R}	$\pi/108A_{\mathrm{R}}$	2	$\sqrt{39}\mathcal{L}$
$(2q, 2p)$	$2A_{\mathrm{R}}$	$\pi/432A_{\mathrm{R}}$	2	$2\sqrt{39}\mathcal{L}$		
Centre–edge	odd–odd	$(3q/2, 3p/2)$	$3A_{\mathrm{R}}$	$2\pi/243A_{\mathrm{R}}$	2	$3\sqrt{37}\mathcal{L}$
	odd–even	$(3q, 3p)$	$3A_{\mathrm{R}}$	$\pi/243A_{\mathrm{R}}$	2	$3\sqrt{7}\mathcal{L}$
	even–odd	$(3q, 3p)$	$3A_{\mathrm{R}}$	$\pi/243A_{\mathrm{R}}$	2	$3\sqrt{7}\mathcal{L}$

Using these results, we get for $L < \min_\alpha(L_{\max,\alpha}/\pi) = \sqrt{2\mathcal{E}/37\sqrt{3}\pi}$,

$$\Delta_3(L, \mathcal{E}) = \frac{28}{27}\frac{L}{15} + \frac{1}{3^{7/4}2^{1/2}\pi^{3/2}}\frac{L^2}{\mathcal{E}^{1/2}} \tag{15}$$

and for $L \gg \max_\alpha(L_{\max,\alpha}/\pi) = \sqrt{2\sqrt{3}\pi\mathcal{E}/3}$,

$$\Delta_3(L, \mathcal{E}) = \frac{\sqrt{\frac{\sqrt{3}}{6}}[3276 + 728\sqrt{3} + 117\sqrt{7} + 63\sqrt{13} + 312\sqrt{21}]}{14\,742}\frac{\mathcal{E}^{1/2}}{\pi^{3/2}} + \sqrt{\frac{\mathcal{E}/2}{3^{5/2}\pi^5}}. \tag{16}$$

From the expressions and examples discussed above, one can clearly see that there is a universal trend of $\Delta_3(L)$ with L for integrable and pseudo-integrable billiards. More precisely, for $L < \min_\alpha(L_{\max,\alpha}/\pi)$, the rigidity is very well approximated by $L/15$, and for $L \gg \max_\alpha(L_{\max,\alpha}/\pi)$ it saturates with a crossover connecting these two limits smoothly. The extent of the crossover region is given by the difference between $\min_\alpha$ and $\max_\alpha$ of $L_{\max,\alpha}$, or in other words, depends on the spectrum of lengths of shortest periodic orbits over α. Non-universal aspects, thus, arise due to non-trivial classification depending upon the degree of tessellation of the invariant surface in terms of a system-specific fundamental region [4]. For instance, in IRB, tessellation is complete and there is only one class of bands

($\alpha = 1$); the crossover region is expected to be of lesser extent—a fact fully corroborated by the numerical experiments. In the GB, there is a barrier (gap-to-barrier ratio is unity) in a rectangle which gives rise to a periodic untessellated arabesque in terms of which the classification is facilitated, the number of bands here is seven. Similarly for the RHB, the number of bands is eighteen. Importantly, it should be noted that the value of L at which the spectral rigidity deviates from the Poisson value of $L/15$, and the value at which saturation sets in, depends upon the lengths of the shortest periodic orbits distributed over various classes admissible in a given system. Indeed, this is the fundamental source of non-universality.

Let us discuss the numerical results of various pseudo-integrable billiards. Most of the studies have been on rhombus billiards [7], square torus billiards and its generalizations [6] and singular billiards [9]. Analysis of singular billiards was carried out and one understands the level spacing statistics [20]. However, there are recent studies [12] which are still unexplained. Study of the two-level cluster function (in particular $\Sigma^2(L)$) does not give the GOE result [21] although the level spacing is GOE raising, a question currently beyond explanation. Therefore, we concentrate on an explanation of the results for non-singular systems.

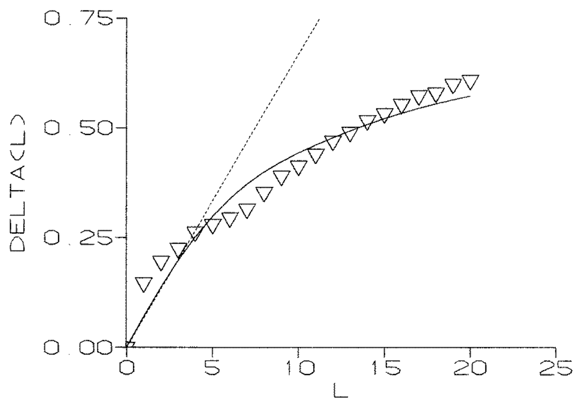

Figure 1. Δ_3 statistic for $\pi/3$-rhombus billiards. The full curve represents our results for $\mathcal{E} = 350$, the triangles represent the result of [7] and the broken line represents the Poissonian.

Perhaps the paradigm PIB is also RHB [7, 8]. In both these studies, one can observe that the rigidity is intermediate to Poisson and GOE. From our analysis, taking the energy and parameters from these numerical works, it turns out that deviation from $L/15$ would occur at $L \sim 1$ and 2, respectively. We illustrate this in figure 1, where we compare our analytical result with that of the numerical work [7]; the agreement is clearly evident. Oscillations in the numerical result about our curve may be due to a different averaging procedure. The crossover values are also correctly predicted by our analysis. We show the behaviour of Δ_3 for small valus of L in figure 2, where deviation from $L/15$ is evident. In figure 3 we show δ_3 for complete range of L, where the crossover region and the saturation can be seen. Since numerical results are not available for higher energy and for a larger range of L, the saturation cannot be seen clearly in the numerical experiments. It is, therefore, desirable to carry out extensive numerical work for higher energy and for a larger range of L. The formulae (7), (8) provide guidelines for choosing the appropriate number of levels to bring out all the salient features of the systems discussed above. Our analysis also explains the results of [6] where one gets $L/15$ for very small values of L and there is

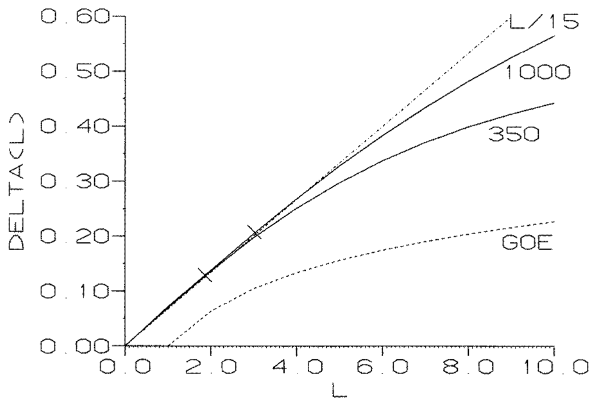

Figure 2. Δ_3 statistic for $\pi/3$-rhombus billiards. The full curves represent rigidity for 350 and 1000 levels: in our analysis $\mathcal{E} = 350$ and 1000, respectively. $L = \min_\alpha(L_{\max,\alpha})$, where rigidity deviates from $L/15$ are shown by \ on the respective curves.

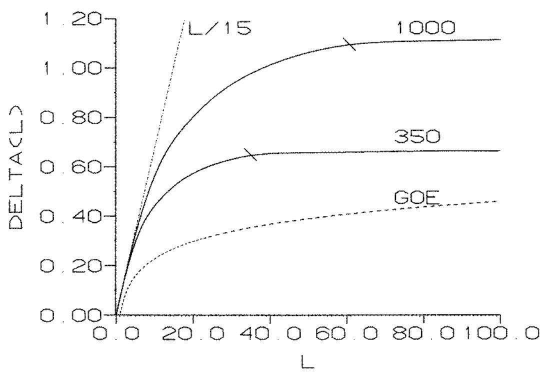

Figure 3. Δ_3 statistic for $\pi/3$-rhombus billiards. The full curves represent rigidity for 350 and 1000 levels: in our analysis $\mathcal{E} = 350$ and 1000, respectively. $L = \max_\alpha(L_{\max,\alpha})$ above which the start of saturation is shown by \ on the respective curves.

a saturation regime. Unfortunately, because of constraints over levels available, the belief of an intermediate behaviour between that of Poisson and GOE has been pursued for quite some time. Our analysis clearly reveals that such a behaviour does not exist and that spectral rigidity never becomes GOE (it is a fundamentally different objective to fit a curve to the GOE result when one is seeking for a theory).

The occurrence of periodic orbits in the bands is a likely reason for the slow rise of $\Delta_3(L)$ in the large L region and overall stronger fluctuations than the GOE result. A recent result on stadium billiards also indicates this possibility [22]. In [22], $\Delta_3(L)$ is shown to be rising well above the GOE curve if the contribution of the bouncing ball modes is taken into account. In chaotic systems like this (also, e.g. the Sinai billiard) the analysis of bands can be carried out using the above theory and it is expected that there exists a departure from GOE as well as a rise in spectral fluctuations after some L decided by the length of the periodic orbits in the band. Recently, non-genericity of the rigidity arising from banded orbits has been discussed for the stadium billiards [23].

Finally, we comment here that the theory and results discussed above are not specific to the examples we have considered, for example the same theory will hold good for generic pseudo-integrable systems, e.g. one can vary either barrier length or the position of the barrier in GB which will result in different classification, but will follow the same treatment as above. Work in this direction is in progress.

In conclusion, we have developed a theory for the Δ_3-statistics for systems in which periodic orbits of the marginal stability (in bands) occur, which gives us a formula for the rigidity from which Poisson and non-Poission results follow in a natural way. Answers to the basic questions we have asked above are as follows. (i) The levels of PIB are uncorrelated and mimic a Poission process for $L < \min_\alpha(L_{\max,\alpha})$ which depends on the shortest periodic orbit of a given system, hence a non-universal value. This condition also stipulates the minimum number of energy levels that one should consider in the numerical experiment to observe this effect. (ii) For $L \gg \max_\alpha(L_{\max,\alpha})$, which depends mainly on the longest of the shortest PO among the different classes, the spectral rigidity saturates to a non-universal value. (iii) The fluctuation properties of PIB and IB differ essentially in the extent of transition region. In IB the transition region will be of less extent, since there is only one class of PO; deviation from Poission and saturation is determined by the same PO (i.e. the shortest one). In PIB because large number of classes of PO are present, all shortest PO among the different classes play important roles in determining the shape and extent of the crossover region. In a similar manner, subsequently, the general two-level cluster function and the form factor have recently been found [24].

We acknowledge our immense debt of gratitude to Akhilesh Pandey, without his critical remarks the work would not have taken the present form.

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